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VALUE OF INFORMATION WITH SEQUENTIAL FUTURES MARKETS[1]

BY JERRY GREEN

The effects of an improvement in information on the efficiency of risk-bearing are studied under various systems of incomplete markets. With sequential futures markets for uncontingent delivery, the welfare effects are indeterminate in sign, except under special circumstances. In the presence of options markets, however, an improved information structure is almost surely beneficial.

1. INTRODUCTION

Value of Information in Economic Models

ALTHOUGH THE VALUATION of information structures has been a topic of much concern in statistical decision theory, the economic literature devoted to this problem is fragmentary.[2] In this paper I will first review what is known about the efficacy of improving the quality of publicly disseminated information in models of general economic equilibrium. Particular attention will be paid to the relationship between the structure of markets—their timing and the nature of the contracts traded—and the sequential process through which information is revealed.[3]

I will then consider a simple partial equilibrium model[4] in which the relationship between the ordering of information structures in the decision-theoretic sense and their valuation to economic agents can be studied in more detail. The

[1] This research was supported by National Science Foundation Grants SOC 71-03803 and APR 77-06999 to Harvard University. The author is grateful to Elon Kohlberg for several helpful discussions.

[2] Hirschleifer is responsible for having properly emphasized the potentially detrimental effect of improved information in [**6**]. In Hirshleifer [**7, 8**] the value of alternative market structures is explored for the case of two states of nature. One of the market structures explored in the former is the same as that treated in Sections 3–6 of this paper. However, there are several differences between the present study and that approach. First, Hirshleifer was primarily concerned with the effects of differences in prior beliefs on speculative behavior (see also Feiger [**5**]), whereas we will deal exclusively with the case of a common prior distribution for all agents. Second, these papers do not address the question of improving a given information structure which is the primary focus of this study. Finally, the case of two states of nature has some special characteristics that do not generalize. We deal with a more general state space; but our analysis will be more restrictive than Hirshleifer's in some other respects. A related paper is Marshall [**16**].

The welfare effects of improving the information structure are studied by Bradford and Kelejian [**3, 4**]. These are similar in spirit to our approach, but are concerned with welfare gains from improved intertemporal resource allocation rather than from a superior allocation of risk bearing. The results also depend heavily on certain parametric specifications. See also Hayami and Peterson [**14**].

[3] This follows the general outlines set out in Radner [**19**] but is specialized to the case of common information. We also treat the case in which the market structure is coarser than the information structure, which is the primary reason for the complexity of the present analysis.

[4] This model was studied in some special cases in Green [**11, 12**]; the present paper is more general in its assumptions and also offers a comparison of alternative market structures that was not given in these papers.

McKinnon [**18**] employs this model, with trade only after the information is received, to compare the efficacy of buffer stock policies with other possibilities for risk avoidance.

principal conclusions are that in a model with futures markets for unconditional delivery of the commodity, an improvement of the information structure in the decision theoretic sense generally will not coincide with a desirable change from the economic standpoint. Sufficient conditions for such a coincidence of rankings are given, but they are unlikely to be satisfied in practice. If options trading is allowed, and if there are only finitely many states of nature, then the two rankings will coincide except for rare, negligible, circumstances. But with a continuum of states of nature this result may not persist and there are some remaining open questions in this regard.

2. INFORMATION STRUCTURES

The uncertainty faced by a decision-maker is represented by his lack of knowledge of a parameter $\theta \in \Theta$. We assume that Θ is a finite set

$$\Theta = \{\theta_1, \dots, \theta_m\}.$$

There are several ways of formalizing the concept of an information structure. The more usual one is to give a set of possible observations Y and m probability measures on Y, π_i, $i = 1, \dots, m$. The interpretation is that if θ_i is the true state, then the observation is distributed according to π_i. It is denoted (Y, π) for brevity.

The decision-maker's problem is to choose an action $a \in A$ so as to maximize the expected value of his utility

$$u : \Theta \times A \to \mathbb{R}.$$

The selection of a is made after $y \in Y$ is observed, so the relevant expectation for each y is the posterior belief. This depends on the prior probabilities

$$r = (r_1, \dots, r_m)$$

according to Bayes theorem:

$$\lambda(\theta_i | y) = \frac{\pi_i(y) r_i}{\sum_k \pi_k(y) r_k}.$$

The attained level of expected utility, viewed *ex ante*, depends upon u, r, and the information structure (Y, π). It is denoted $U(u, r, (Y, \pi))$. Information structures are partially ordered by the criterion that

$$(Y, \pi) \circledcirc (Y', \pi')$$

if and only if

$$U(u, r, (Y, \pi)) \geqq U(u, r, (Y', \pi')) \quad \text{for all } u \text{ and all } r > 0. \tag{2.1}$$

Blackwell's theorem [5] provides a set of alternative, equivalent, comparisons of information structures that are simpler to check than (2.1). Let us assume that the

[5] See Blackwell [**1**], Blackwell and Girshick [**2**], Marschak and Miyasawa [**15**], or McGuire [**17**].

information structures (Y, π) and (Y', π') involve sets of observations with n and n' elements respectively. (This can be generalized directly.) Blackwell's theorem can be stated as:

$$(Y, \pi) \circledcirc (Y', \pi')$$

if and only if there is a Markov matrix $B = (b_{jj'})_{n \times n'}$ such that

$$\Pi' = \Pi B \tag{2.2}$$

where

$$\Pi' = (\pi_i'(y_{j'}'))_{m \times n'} \quad \text{and} \quad \Pi = (\pi_i(y_j))_{m \times n}.$$

This theorem has two principal implications for our purposes. First, (2.1) has been converted into a constructive criterion involving linear inequalities (because $b_{jj'} \geqslant 0$ is required), and is therefore easier to verify. Second, this criterion is independent of the prior, r. For any fixed $r > 0$, the ranking induced by

$$U(u, r, (Y, \pi)) \geqslant U(u, r, (Y', \pi')) \quad \text{for all } u \tag{2.3}$$

will be identical with $\circledcirc$.

When the prior is fixed, there is an alternative formalization of an information structure. This is the one that will be utilized below. An *information structure* (for (Θ, r)), $(X, \mu, \mathscr{S})$, is a set X, a measure μ on $\Theta \times X$ and a partition $\mathscr{S}$ of X. A generic subset of X is denoted S. The marginal distribution of μ on Θ is constrained to be r. The interpretation is that X is a set of potential observations, x, that are statistically related to θ as described by μ. However, these best possible observations are not actually received; only the set $S \in \mathscr{S}$ is perceived and can be used to condition the choice of an action.

Clearly any (Y, π) can be reformulated as $(X, \mathscr{S}, \mu)$ and vice versa. However, to compare (Y, π) and (Y', π') in the formulation (2.1) it is necessary to utilize the same set X and measure μ.

It has been shown (Green and Stokey [**13**]) that if $(Y, \pi) \circledcirc (Y', \pi')$, then there exists (X, μ) and two partitions of X, $\mathscr{S}$ and $\mathscr{S}'$, such that $(X, \mu, \mathscr{S})$ is equivalent to (Y, π), $(X, \mu, \mathscr{S}')$ is equivalent to (Y', π'), and $\mathscr{S}$ refines $\mathscr{S}'$.

Throughout this paper we deal with information structures specified as $(X, \mu, \mathscr{S})$, and we consider comparisons of partitions (partially) ordered by refinement. By virtue of the theorem just cited we can interpret results applicable to any pair of partitions ordered by refinement as being valid for any information structures ordered by Blackwell's criterion.

3. IMPROVING INFORMATION, STATISTICAL DECISION THEORY, AND ECONOMIC MODELS OF EQUILIBRIUM

In statistical decision theory a *decision problem* is given by a set of possible actions A and a utility function $u: \Theta \times A \to \mathbb{R}$. The *value* of the problem $v(u, \mathscr{S}, \mu)$ is given by $E_{S \in \mathscr{S}} \max_{a \in A} E_{\theta|S} u(\theta, a)$. Obviously the value of a problem increases when $\mathscr{S}$ is refined.

In economic models there are two additional buyers of complexity. Both arise from the interaction between the endogenous variables determined by the model and the economic environment. First, the set of feasible actions may be a function of endogenous variables, as for example is the budget set of a consumer dependent on prices. Second, the actual payoff may depend on these endogenous variables as well as on θ and a. Fluctuations in the value of long-term financial assets is an example.

We will consider the case of identical prior beliefs and information across agents. All agents' posterior beliefs, given S, are given by the mapping $\lambda(\cdot|S)$ defined by:

$$\lambda(\theta|S)=\frac{\mu(\{\theta\}\times S)}{\mu(\Theta\times S)}.$$

Let Δ be the set of all probability vectors. A *decision structure* $\phi=(\phi_A, \phi_U)$ is the formal description of the dependence of an economic agent's decision problem on the common posterior. It is specified by the pair of functions:

$$\phi_A:\Delta\to\mathscr{A}$$

and

$$\phi_U:\Delta\to\mathscr{U},$$

where $\mathscr{U}$ is the space of all utility functions and $\mathscr{A}$ is the space of all subsets of A. For any $\lambda\in\Delta$, $\phi_A(\lambda)$ is the subset of A to which choice is restricted when the information leads to the posterior λ. Similarly $\phi_U(\lambda)$ is the induced utility function over pairs (θ, a).

The value of an information structure depends upon the decision structure to which it is applied. Let $V(\phi, u, \mathscr{S}, \mu)$ be given by

$$E_S \max_{a\in\phi_A(\lambda(\cdot|S))} E_{\theta|S}\phi_U(\lambda)(\theta, a).$$

It is easy to see that, for some ϕ, $V(\phi, u, \mathscr{S}, \mu)$ might actually decrease when $\mathscr{S}$ is refined. Let us define the partial ordering of partitions $>_\phi$ by $\mathscr{S}>_\phi\mathscr{S}'$ if and only if $V(\phi, u, \mathscr{S}, \mu)\geqslant V(\phi, u, \mathscr{S}', \mu)$ for all pairs (u, μ). The primary goal of this research is to ascertain whether $\mathscr{S}$ refines $\mathscr{S}'$ implies $\mathscr{S}>_\phi\mathscr{S}'$ when ϕ is derived from the endogenously determined prices in a model of economic equilibrium.

More specialized related goals are to study the partial ordering $>_{\phi,\mathscr{M},\mathscr{U}}$ where $\mathscr{U}$ is a set of utility functions, and $\mathscr{M}$ is a set of measures on $\Theta\times X$ with the same marginal distributions over Θ, which is defined by $\mathscr{S}>_{\phi,\mu,\mathscr{U}}\mathscr{S}'$ if and only if

$$V(\phi, u, \mathscr{S}, \mu)\geqslant V(\phi, u, \mathscr{S}', \mu)$$

for all $u\in\mathscr{U}$, and all $\mu\in\mathscr{M}$. In particular applications $\mathscr{U}$ and $\mathscr{M}$ may be restrictions to particular parametric forms or to sets of utilities and measures having some specified qualitative characteristics.

4. VALUE OF INFORMATION IN GENERAL EQUILIBRIUM MODELS

Before turning to the analysis of a particular partial equilibrium model which will be the main focus of our attention, we pause to survey the current state of the problem in general equilibrium models. This will place the later analysis in the proper perspective. One or two of the results of this section may be new, but most are well known.

The value of improving the information structure in a general equilibrium system depends on two principal factors: the timing of markets compared with the timing of the informational structure, and the presence or absence of a complete system of futures markets for contingent trade.

We discuss the case of pure exchange and consider only information structures and priors that are the same for all agents. Individuals are denoted $i = 1, \ldots, I$; their endowments are $w_i(\theta)$; their trades are t_i; and their consumptions are $\xi_i = w_i + t_i$. Utility will be assumed to be state independent and given by the von Neumann-Morgenstern representation $u_i(\xi)$. Some of our results will depend on this restriction, but since it will be maintained in the partial equilibrium model of Sections 5–7 we have assumed it here in order to make the analyses more readily comparable.

A. *Complete markets for trade conditional on (S, θ) held before information is revealed, with consumption only after θ is revealed.*

In this case, it is well known that trade dependent on S would be irrelevant and a Pareto optimal resource allocation would be attained with trade conditioned only on θ. The information structure therefore has no bearing on the allocation of resources.

B. *Complete markets for trade conditional on (S, θ) held before information is revealed, with consumption at both the date information is revealed and when θ is revealed.*

In this model the state θ is revealed in two stages. The information structure $\mathcal{S}$ serves to limit the set of possible states that can occur with positive probability. That is $\{\text{supp } \lambda(\cdot|S)\}_{S \in \mathcal{S}}$ is a partition of Θ. We denote this partition by $\mathcal{T}$.

From the work of Radner [**19**] we know that the equilibrium of this system is Pareto efficient among all allocations restricted to be measurable with respect to $\mathcal{T}$ at the information date. Clearly, therefore, a refinement of the information structure expands the feasible set and cannot be detrimental to all agents at the new equilibrium. On the other hand, there is no guarantee that every agent is made better off at the equilibrium with better information than at the original one.

C. *Complete markets for trade conditional on θ held after the revelation of S. Consumption only after θ is revealed.*

Here, the price system depends on the set S. Unlike the case in which markets meet before S is revealed, there is the potential for information to be harmful. (This has been pointed out in Hirschleifer [**6**] but to my knowledge the precise general statement has not been given previously, nor has it been emphasized that the case of "no information" has some special properties.)

THEOREM: *Let $\mathcal{S}_0 = \{X\}$ and let $\mathcal{S}$ be any other information structure. Then for some agent the attained level of expected utility must be at least as high in the equilibrium with $\mathcal{S}_0$ than with $\mathcal{S}$. If utilities are strictly concave then it must be higher.*

PROOF: Let $\xi_i(S, \theta)$ and $\xi_i(X, \theta)$ be the allocations attained under $\mathcal{S}$ and $\mathcal{S}_0$ respectively. By feasibility we have that

$$\sum_i \xi_i(S, \theta) = \sum_i \xi_i(X, \theta) = \sum_i \omega_i(\theta)$$

for every $S \in \mathcal{S}$, and every $\theta \in \Theta$.

For each θ we have by concavity that

$$E_{S|\theta} u_i(\xi_i(S, \theta)) \leqslant u_i(E_{S|\theta}\xi_i(S, \theta)) \tag{4.1}$$

for all i (with strict inequality if the utility is strictly concave).

The allocation $(\xi_i(X, \theta))_{i=1,\dots,I}$ is Pareto undominated in expected utility, by any feasible allocation varying only with θ. In particular, there must be some i for which

$$E_\theta u_i(E_{S|\theta}\xi_i(S, \theta)) \leqslant E_\theta u_i(\xi_i(X, \theta)) \tag{4.2}$$

since $(E_{S|\theta}\xi_i(S, \theta))_{i=1,\dots I}$ is such a feasible allocation.

Integrating both sides of (4.1) with respect to θ and combining the result with (4.2) we have that, for some i,

$$E_{\theta,S} u_i(\xi_i(S, \theta)) \leqslant E_\theta u_i(\xi_i(X, \theta))$$

(with strict inequality given strictly concave utilities).

REMARK: It is not true that the expected utility for some agent must fall when an arbitrary information structure $\mathcal{S}$ is refined to $\mathcal{S}'$. The reason is as follows: We may write, paralleling (4.1), that

$$E_{\theta,S} E_{S'|S} u_i(\xi_i(S', \theta)) \leqslant E_{\theta,S} u_i(E_{S'|S}\xi_i(S, \theta)) \tag{4.3}$$

for all i.

Although for each S there is some i such that

$$E_{\theta|S}u_i(E_{S'|S}\xi_i(S', \theta)) \leqslant E_{\theta|S}u_i(\xi_i(S, \theta)), \tag{4.4}$$

the agent i is potentially different for each S. Therefore upon integrating (4.4) with respect to (θ, S) the inequality

$$E_{\theta,S}u_i(\xi_i(S', \theta)) \leqslant E_{\theta,S}u_i(\xi_i(S, \theta)) \tag{4.5}$$

may not hold for any i.

Such phenomena in second-best welfare analysis away from the global optimum are often encountered.

D. *Incomplete markets held before S is revealed, with consumption only after θ is revealed.*

The allowable trades must be measurable with respect to a partition, $\mathscr{M}$, of Θ which is coarser than the partition of Θ into one-element sets. In such a system the equilibrium is known to be Pareto optimal relative to the set of all allocations which are $\mathscr{M}$-measurable. However, after θ is revealed and trades are consummated, there is a further incentive to trade and markets would reopen if it is feasible to do so. If such a reopening of markets is perceived *ex ante* the initial equilibrium would be disturbed because of speculative motives, and the presumption of its optimality is no longer valid. A temporary equilibrium theory would have to be developed along the lines of Green [**10**] to admit the possibility of speculative trading.

E. *Incomplete market after information is revealed, with consumption only after θ is revealed.*

In this case the argument of Section C breaks down because $\xi_i(S, \theta)$ will not in general be $\mathscr{M}$ measurable. Although for each S

$$E_{S|\theta}u_i(\xi_i(S, \theta)) \leqslant u_i(E_{S|\theta}\,\xi_i(S, \theta))$$

the allocation $E_{S|\theta}\xi_i(S, \theta)$ may not be feasible given $\mathscr{M}$ because the trades required to sustain it, $E_{S|\theta}t_i(S, \theta)$ may not be $\mathscr{M}$ measurable. This raises the possibility that $\mathscr{S}$ may dominate $\mathscr{S}_0$. Examples to this effect have been given in Green [**11**].

F. *Incomplete markets both before and after information is revealed, with consumption only after θ is revealed.*

Because the randomness of the price system might cause detrimental effects on individuals' welfare compared to a stable price system, there would be an incentive to trade before the information is announced. With complete markets for trade conditional on θ and trade prior to information we would have a market structure such as in A above; and as noted there, the information structure would be irrelevant to allocation and the markets held after the information is revealed would become vestigial. With incomplete markets, however, an incentive to revise trades made prior to the information would be present. Both rounds of trading would be active unless some form of impediment precluded them.

Speculative motives in the prior round of trading would determine the allocation achieved. The welfare analysis of such a system with sequential, incomplete, futures markets is highly complex and it is doubtful that general results could be obtained. Nevertheless this market structure is in some sense the most natural (inevitable?) when a complete contingent claims market cannot be established. It is, indeed, a representation of reality in many markets—for example bonds, foreign exchange, and commodity futures trading. The partial equilibrium analysis that follows is therefore based on this structure: trade, both before and after information is revealed, for a commodity to be delivered independent of the state of nature that is realized.

5. A PARTIAL EQUILIBRIUM MODEL

A. *Market Structure and Informational Conditions*

We consider a partial equilibrium model involving a single commodity. The production and demand conditions are uncertain. Although there are many producers whose output levels are jointly distributed random variables, we will consider the welfare of only one producer. His decisions about productive inputs are assumed to be taken prior to the events and activities described in this model, and moreover are assumed to be nonresponsive to changes in his economic environment. The analysis is concerned exclusively with the way in which the market and the information structures interact and enable this producer to reduce the risk he faces.

The producer is assumed to have no residual demand for his output. His behavior is governed by a von Neumann-Morgenstern utility function defined over the level of his final wealth. This wealth is the sum of the value of his output plus any profits or losses he has made by trading futures contracts.

The nature of informational considerations in this model is as follows: When the underlying uncertainties of production and demand have been resolved, there is a final spot market on which the available output is traded. The equilibrium price is denoted p. The product of p with the producer's realized level of output is the value of his output and is denoted by v. Since p and v are both functions of the underlying state of nature, $\theta \in \Theta$, whose distribution is exogenous, they are regarded by the producer as jointly distributed random variables. We will sometimes write $p(\theta)$ and $v(\theta)$ explicitly to emphasize this dependence, but usually the parameter θ is suppressed without fear of confusion.

At some time before the uncertainty is resolved there is an observation, $S \in \mathcal{S}$, received in common by all the economic agents in the system, that is related to the state of nature in a statistical sense, as described in Section 2.

Let us now consider the structure of futures markets. Throughout this study a futures contract is a contract for (uncontingent) delivery of the commodity at a specified price. This price is the equilibrium price of such a contract at the date the futures market is open, which may be either before or after S is revealed, as described in Section 4.

The market before S is revealed is called the *prior* market, and the market after S is revealed is called the *intermediate* market—to indicate that it comes between the information and the resolution of the uncertainty. Alternative market structures consist of specifying whether either, or both, of these futures markets are allowed to be active.

Holding the structure of the futures markets fixed, we ask whether an improvement in the quality of the information structure, in the sense of Section 2, necessarily increases the producer's expected utility. In particular we show that the value of any information structure compared to "no information" is always positive when both prior and intermediate markets are operative, although it may be negative when there is only an intermediate market.[6] However, improving an existing information structure does not in general lead to a higher expected utility. Sufficient conditions to insure "monotonicity" of the producer's expected utility in the quality of the information are given, but they are very restrictive in nature.

B. *Futures Markets Equilibration*

The issues above can best be approached under a simplified assumption regarding the equilibration of future markets.

ASSUMPTION: Every futures market equilibrates at a price equal to the conditional mean price on the final spot market, given the information available.

That is, if we are considering a futures market before the information is revealed, then $p^0 = E_\theta p$ is the equilibrium price. If the observation S has already been received, then $p^1(S) = E_{\theta|S}p$ is the equilibrium. This assumption is justified if there are risk neutral arbitrageurs or if this market is small and independent of other risks in the economy.

The von Neumann-Morgenstern utility function is denoted u, and the agent's final wealth is denoted w.

If a futures contract is purchased, the profit from holding it to the final date is the difference between the price on the spot market and the futures price.

Let z^0 be the purchases on prior futures market, held until the intermediate futures market and then sold. Let $z^1(S)$ be the purchases on the futures market after information, S, has been observed.

If both types of futures markets are operative, the individual's decision consists of a choice of z^0 and a planned choice of $z^1(S)$ for every $S \in \mathcal{S}$. This strategy leads to final wealth

$$w = v + z^0(p^1(S) - p^0) + z^1(S)(p - p^1(S)).$$

[6] For an example in which this occurs, see Green [**12**].

The joint distribution of (θ, x), together with $\mathscr{S}$ and the functions $p(\theta)$ and $v(\theta)$ determine the distributions of w. If only the prior futures market is open, and if z denotes the level of purchases, then

$$w = v + z(p - p^0).$$

If only the intermediate futures market is open, then

$$w = v + z^1(S)(p - p^1(S)).$$

Just as in Section 4F we see that, given an information structure, the agent can only gain from the existence of an opportunity to trade on futures markets at both the prior and intermediate dates. In the absence of a prior market he is effectively restricted to choose $z^0 = 0$, and without an intermediate market he must take $z^0 = z^1(S) = z$ for some z. Since any restriction on the class of attainable random payoffs w can only be harmful, all agents would prefer to have both rounds of trading available.

6. VALUE OF CHANGING THE INFORMATION STRUCTURE

A. *Is Some Information Better Than None?*

In Green [**12**] it was shown that with only an intermediate market, a given information structure may or may not be superior to no information at all. The reason is that in improving the information structure the system of prices at which the agent can trade on the intermediate market, $\{p^1(S); S \in \mathscr{S}\}$ becomes more variable with respect to S. This variability may more than offset the direct beneficial effect of superior information that allows the individual to make better use of the futures markets to mitigate risk. In the present paper we explore whether the presence of a prior market on which the price is necessarily independent of the information may enable the agent to offset this price fluctuation.

It is easy to see that any information structure allows the agent to achieve at least the expected utility attainable under no information, when both rounds of trading are active. Without information the prior and intermediate rounds of trading are really the same, that is, $p_1(S) \equiv p^0$. Let the level of contracts held after the intermediate round be z^1. To achieve the same distribution of final wealth in a system with information the agent could always choose $z^0 = z^1(S) = z^1$, for all S. Thus, information can only be beneficial, vis à vis a "no information" situation.

B. *Is Any Improvement in the Information Structure Beneficial?*

A Positive Example

The situation becomes more complex when we consider improvements of an existing information structure that is already better than "no information." To approach this problem we proceed via an example.

Let us take

$$X = \mathbb{R} \quad (\text{with } s \in X \text{ denoting a typical element}),$$

$$\Theta = \mathbb{R}^2, \quad p(\theta_1, \theta_2) = \theta_1, \quad v(\theta_1, \theta_2) = \theta_2,$$

μ trivariate normal with mean (0, 0, 0) and variance covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{pp} & \sigma_{pv} & \sigma_{ps} \\ \sigma_{pv} & \sigma_{vv} & \sigma_{vs} \\ \sigma_{ps} & \sigma_{vs} & \sigma_{ss} \end{pmatrix}. \tag{6.1}$$

That is, we have identified p and v with the first and second factor spaces of Θ and can then consider them directly as the underlying random variables.

The mean of this distribution can be taken to be (0, 0, 0) without loss of generality, for the following reasons: The price, p, is only relevant relative to its deviation from the mean, and thus it can be translated to have mean zero. The scaling of s is arbitrary; all that is really relevant are the conditional joint distributions of (p, v) for each s.

As far as v is concerned, it will soon become clear that any policy $(z^0, (z^1(S))$ results in a payoff having a normal distribution, or a mixture of normal distributions, with mean Ev. In such a situation, every risk-averter would want to minimize the variance of the payoff and would choose the same policy. Thus Ev can be normalized to zero.

This point bears a bit more emphasis since it is primarily responsible for the simplicity of the results in this example. Risk averters may have diverse attitudes towards the tradeoff between risk and return. Such an agent whose mean wealth changes, generally will have a different preference over risky assets as a result. However, the special feature of this model is that all policies affect the distribution of returns but not the mean. Therefore the optimal $z^1(s)$ must be such as to minimize the variance of $v + z^1(s)(p - p^1(s))$ for every s, no matter what value of z^0 was selected, because the return $z^0(p^1(s) - p^0)$ is nonstochastic at the date when the trade $z^1(s)$ is executed.

Because of this property, $z^1(s)$ depends only on the conditional variance-covariance matrix of v and $p - p^1(s)$ which is given by

$$\begin{pmatrix} \sigma_{pp} - \dfrac{\sigma_{ps}^2}{\sigma_{ss}} & \sigma_{pv} - \dfrac{\sigma_{ps}\sigma_{vs}}{\sigma_{ss}} \\ \sigma_{pv} - \dfrac{\sigma_{ps}\sigma_{vs}}{\sigma_{ss}} & \sigma_{vv} - \dfrac{\sigma_{vs}^2}{\sigma_{ss}} \end{pmatrix}. \tag{6.2}$$

Note that this is independent of s, a property of the joint normality assumption. Therefore $z^1(s)$ is in fact independent of s and is given by

$$z^1(s) = \frac{\sigma_{ps}\sigma_{vs} - \sigma_{pv}\sigma_{ss}}{\sigma_{pp}\sigma_{ss} - \sigma_{ps}^2}. \tag{6.3}$$

To minimize the overall variance of w, one can compute directly that the optimal trade in the prior market is

$$z^0 = \frac{-\sigma_{vs}}{\sigma_{ps}}. \tag{6.4}$$

The variance of the optimum random payoff is

$$\sigma_{vv} - \frac{\sigma_{vs}^2}{\sigma_{ss}} - \frac{(\sigma_{ps}\sigma_{vs} - \sigma_{pv}\sigma_{ss})^2}{\sigma_{ss}(\sigma_{pp}\sigma_{ss} - \sigma_{ps}^2)}. \tag{6.5}$$

First let us note that the formula for the variance at the optimum (6.5) is consistent with the result that no information is always dominated by some information. The absence of information in this example is equivalent to $\sigma_{ss} = \infty$, since this implies that any observation s is associated with a conditional distribution of (p, v) that is identical to its unconditional distribution. Setting $\sigma_{ss} = \infty$, (6.5) becomes

$$\sigma_{vv} - \frac{\sigma_{pv}^2}{\sigma_{pp}} \tag{6.6}$$

and it can be computed directly from (6.5) and (6.6) that the variance of the payoff is reduced when σ_{ss} is finite.

More generally, the derivative of (6.5) with respect to σ_{ss} can be shown to be always positive. Thus an improvement in the quality of information that consists of reducing the variance of s is necessarily beneficial.[7] Not all improvements in the initial structure of information can be represented in this way. However, any improvement with the property that the posterior distribution of (p, v) was a bivariate normal with conditional variance-covariance matrix independent of S can be represented as a refinement of an original partition on $\Theta \times X$ if suitably large X space were chosen.

Although it has been shown that a class of improvements of the information structure are beneficial, it has not been demonstrated that an arbitrary refinement will be beneficial.

C. *A Negative Example*

In the example above we saw that the sequential market structure enabled the producer to obtain the benefits of superior information in the intermediate market, due to a higher conditional correlation between p and v, while mitigating the increased risks of induced price fluctuations by taking an appropriate position in the prior market. This positive result depends on several special features of the normal distribution. To illustrate, and to prepare the ground for the theorem of the next section, we present another example.

[7] When σ_{ss} is decreased the underlying space X can be reinterpreted as $\mathbb{R}^2$ and μ can be thought of as a jointly normal distribution over $\mathbb{R}^4$. Then the coarser information structure is the set of cylinder sets over the first component of X and the better one is the partition of X into one-element sets.

This example has the striking characteristic that the improvement in the information structure will be detrimental to the utility of *every* risk averter. Under any criterion it will be ranked below the coarser structure.

Let Θ and X consist of three discrete points each:

$$\Theta = \{\theta_1, \theta_2, \theta_3\},$$

$$X = \{x_1, x_2, x_3\}.$$

The variables p and v depend on θ according to

θ	p	v
θ_1	1	1
θ_2	2	1
θ_3	$4\frac{1}{2}$	2.

The underlying joint distribution of (θ, x) is given by the entries in the following table:

	x_1	x_2	x_3
θ_1	$\frac{3}{16}$	$\frac{1}{16}$	0
θ_2	$\frac{1}{16}$	$\frac{3}{16}$	0
θ_3	0	0	$\frac{1}{2}$

We consider two information structures $\mathcal{S}$ and $\mathcal{S}'$ given by

$$\mathcal{S}' = \{S_1', S_2'\} = \{\{x_1, x_2\}, \{x_3\}\},$$

$$\mathcal{S} = \{S_1, S_2, S_3\} = \{\{x_1\}, \{x_2\}, \{x_3\}\}.$$

Clearly $\mathcal{S}$ is a refinement of $\mathcal{S}'$.

Under either information structure, the prior market equilibrium is given by

(6.7) $\quad p^0 = 3.$

The conditional mean prices which are the equilibrium prices on the intermediate market are given by

(6.8) $\quad p^1(S_1) = 1\frac{1}{4},$

$$p^1(S_2) = 1\frac{3}{4},$$

$$p^1(S_3) = 4\frac{1}{2},$$

and

(6.9) $\quad p^1(S_1') = 1\frac{1}{2},$

$$p^1(S_2') = 4\frac{1}{2}.$$

Consider the following strategy which can be followed under the poorer information structure, $\mathcal{S}'$:

$$(6.10)\qquad z^0 = -\tfrac{1}{3},$$
$$z^1(S_1') = z^1(S_2') = 0.$$

Using the definition of the payoff this results in a perfectly certain level of final wealth, w, given by $w = 1\frac{1}{2}$. Since all strategies available to the producer are associated with the same mean return, it is clear that all risk averters would choose the strategy above under the information structure $\mathcal{S}'$.

To show that $\mathcal{S}$ is unambiguously worse than $\mathcal{S}'$ for all risk averters, it suffices to demonstrate that there is no strategy that can achieve this riskless payoff. Consider the three points in $\Theta \times \mathcal{S}$, all of which have a positive probability of occurrence, (θ_1, S_2), (θ_2, S_2), (θ_3, S_3). If we are to achieve a payoff of $1\frac{1}{2}$ under all of these circumstances we must have

$$(6.11)\qquad \text{(i)}\quad 1\tfrac{1}{2} = v(\theta_1) + z^0(p^1(S_2) - p^0) + z^1(S_2)(p(\theta_1) - p^1(S_2)),$$
$$\text{(ii)}\quad 1\tfrac{1}{2} = v(\theta_2) + z^0(p^1(S_2) - p^0) + z^1(S_2)(p(\theta_2) - p^1(S_2)),$$
$$\text{(iii)}\quad 1\tfrac{1}{2} = v(\theta_3) + z^0(p^1(S_3) - p^0) + z^1(S_3)(p(\theta_3) - p^1(S_3));$$

or

$$(6.12)\qquad \text{(i)}\quad 1\tfrac{1}{2} = 1 - 1\tfrac{1}{4}z^0 - \tfrac{3}{4}z^1(S_2),$$
$$\text{(ii)}\quad 1\tfrac{1}{2} = 1 - 1\tfrac{1}{4}z^0 + \tfrac{1}{4}z^1(S_2),$$
$$\text{(iii)}\quad 1\tfrac{1}{2} = 2 + 1\tfrac{1}{2}z^0.$$

From (6.12)(iii) we see that $z^0 = -\frac{1}{3}$ and this plainly contradicts (6.12)(i) and (6.12)(ii). This confirms that the effect of improving the information structure on the expected utility of any risk averter would be harmful.

D. *A Sufficiency Theorem*

We now present a theorem which will cast more light on the differences between the two examples of the preceding section. The conditions of the theorem are extremely restrictive. This is indicative of the limited class of cases under which the ordering of information structures by the criterion of refinement, appropriate for the context of statistical decision theory, coincides with the improvement of expected utility for all risk averters in the present economic model.

THEOREM: *Let $\mathcal{S}$ be a refinement of $\mathcal{S}'$ and let $u = -e^{-\gamma w}$, $\gamma > 0$. Then the following conditions imply that the agent will prefer $\mathcal{S}$ to $\mathcal{S}'$:*

(i) $E_{\theta|S}v = \beta E_{\theta|S}p + \beta_0$ *for all* $S \in \mathcal{S}$;

(ii) $E_{\theta|S}v = \beta' E_{\theta|S}p + \beta_0'$ *for all* $S' \in \mathcal{S}'$;

(iii) *the distribution of* $(v - E_{\theta|S'}v, p - E_{\theta|S'}p)$ *conditional on* S' *is independent of* $S' \in \mathcal{S}$.

REMARKS ON PROOF: The proof consists of using conditions (ii) and (iii) together with the form of the utility function to note some properties of the optimal policy under the coarser information structure $\mathcal{S}'$. Then, condition (i) and the fact that $\mathcal{S}$ is a refinement of $\mathcal{S}'$ are used to show that the distribution of the payoff at the optimum under $\mathcal{S}'$ can be dominated by a distribution attainable under $\mathcal{S}$. This method of proof differs from many in decision theory in that there may be members of the family of payoff distributions, including even the optimum attainable under $\mathcal{S}'$, that are not attainable under $\mathcal{S}$.

Note also that the assumptions (i), (ii), and (iii) hold in the case of joint normality and fail in the negative example of the previous section. The restriction to constant absolute risk aversion utilities is not needed in the case of normality, because all risk averters will choose the distribution with the lowest variance. This implies that their trades in the intermediate market will be independent of the realization of S. Without constant absolute risk aversion, these trades will vary and it is their constancy which turns out to be crucial to the result.

PROOF: We first observe that the optimal decision under the information structure $\mathcal{S}'$, $(z^{0'}, z^{1'}(S'))$ is such that $z^{1'}(S')$ is independent of $S' \in \mathcal{S}'$. To see this, note that for any $z^{0'} \in \mathbb{R}$, the problem that would be faced at the intermediate date given S' is

$$\max_{z^1(S')} E_{\theta|S'} \exp\left(-(v + z^{1'}(S')(p - p^1(S')) + z^{0'}(p^1(S') - p^0)\right).$$

This is clearly identical to

$$\max_{z^1(S')} E_{\theta|S'} \exp\left(-(v - E_{\theta|S'}v) + z^1(S')(p - p^1(S'))\right)$$

which, by virtue of (iii), is the same problem for each value of $S' \in \mathcal{S}'$. Let us denote the common value of the $z^{1'}(S')$ by $z^{1'}$.

The second observation is that $\beta = \beta'$ and $\beta_0 = \beta_0'$ are necessarily satisfied because of the linearity conditions (i) and (ii) and the fact that $\mathcal{S}$ refines $\mathcal{S}'$. We will write their common values as β and β_0, without fear of confusion.

We now proceed to the main part of the proof, which is to show first that the optimal prior trades are $z^{0'} = -\beta$, and then that the sequential trades $(z^0, z^{1'})$ under the information structure $\mathcal{S}'$ are dominated, for all risk averters, by the *same sequence of trades* when $\mathcal{S}$ is the information structure.

Let us fix $S' \in \mathcal{S}$ and consider the collection $\mathcal{T} = \{S \in \mathcal{S} | S \subseteq S'\}$.

The payoff attained under $\mathcal{S}'$ if S' is observed and θ is the realized state of nature is

$$v(\theta) - \beta(E_{\theta|S'}p(\theta) - p^0) + z^{1'}(p(\theta) - E_{\theta|S'}p(\theta)) \tag{6.13}$$

and when $\mathcal{S}$ is the information structure, $S \in \mathcal{T}$ is observed and θ is realized, the payoff is

$$v(\theta) - \beta(E_{\theta|S}p(\theta) - p^0) + z^{1'}(p(\theta) - E_{\theta|S}p(\theta)). \tag{6.14}$$

Given that S' is observed, (6.13) is a random variable that depends only on θ, whereas (6.14) depends on which $S \in \mathcal{T}$ is observed as well. Nevertheless we will consider both payoffs as they depend on S and θ, regarding S and θ as jointly distributed random variables whose distribution is conditional on S'.

We rewrite the payoffs (6.13) and (6.14) as the outcome of a two stage lottery in which, given S', first $S \subseteq S'$ is realized and then θ is realized. Expression (6.13) can be written as

$$\begin{aligned}(6.15)\quad &\{E_{\theta|S}v(\theta)-\beta(E_{\theta|S'}p(\theta)-p^0)+z^{1'}(E_{\theta|S}p(\theta)-E_{\theta|S'}p(\theta))\}+\\ &\qquad\{v(\theta)-E_{\theta|S}v(\theta)+z^{1'}(p(\theta)-E_{\theta|S}p(\theta))\}.\end{aligned}$$

The first bracketed expression is the mean payoff given S, and the second represents the deviation around that mean realized in the second stage of the two stage lottery.

Similarly (6.14) can be rewritten

$$(6.16)\quad \{E_{\theta|S}v(\theta)-\beta(E_{\theta|S}p(\theta)-p^0)\}+\{v(\theta)-E_{\theta|S}v(\theta)+z^{1'}(p(\theta)-E_{\theta|S}p(\theta))\}$$

with the same interpretation of the two terms.

Note that the second parts of (6.15) and (6.16) are identical.

Let us therefore consider the first parts, substituting in the linearity conditions (i) and (ii). From (6.15) we have

$$(6.17)\quad (\beta+z^{1'})(E_{\theta|S}p(\theta)-E_{\theta|S'}p(\theta))+\beta^0+\beta p^0.$$

In (6.16) we have $\beta^0+\beta p^0$.

Therefore the two stage lottery under $\mathcal{S}'$ is riskier than that under $\mathcal{S}$ because the latter has a nonstochastic first stage payoff whereas the former has the same mean $(\beta^0+\beta p^0)$ but is a nondegenerate random variable. Since the second stages are identical, every risk averter would prefer $\mathcal{S}$ to $\mathcal{S}'$, and this applies in particular to the agent in question. *Q.E.D.*

The use of constant absolute risk aversion in this theorem is somewhat hidden and we pause briefly to comment upon it. It is important at only one point, but there it is rather crucial. If we did not have constant absolute risk aversion, the optimum under $\mathcal{S}'$ might correspond to a policy for which the conditional means,

$$E_{\theta|S'}v(\theta)+z^0(E_{\theta|S'}p(\theta)-p^0)$$

are not identical over all $S' \in \mathcal{S}'$. Then, under the refined partition $\mathcal{S}$, the conditional means obtained by choosing the same z^0 would be dispersed even further. Moreover, there might be no choice of z^0 that would make the conditional means, given $S \subseteq S'$, equal to that attained at S'.[8]

Another point should be made before leaving the discussion of this theorem. We showed that the same action plan that was optimal in the case of $\mathcal{S}'$ would

[8] A counterexample to this effect is available from the author. It is omitted here in the interest of conserving space, as it is quite lengthy.

achieve superior results when $\mathscr{S}$ is the information structure. Actually, although the actions are described the same way, they result in different payoffs as a function of θ because the intermediate prices are different.[9] This is properly viewed in the context of Section 2 as a case where the information structure changes the attainable utility level because of induced changes in the endogenous variables. The beneficial effect does not run through its "informativeness" per se. There may, however, be further beneficial effects in this direction if the optimum under $\mathscr{S}$ is not the dominating strategy $(z^{0'}, z^{1'})$ that was used in the proof.

7. VALUE OF INFORMATION IN THE PRESENCE OF OPTIONS MARKETS

The restrictive character of the theorem of the last section derives from the fact that the market structure we have considered imposes constraints on the way in which the producer can hedge his risks. Since all contracts specify delivery unconditionally at the terminal date, the profit or loss from holding futures contracts is necessarily a linear function of the final spot price and the conditional mean prices. The more complex patterns of profit that would allow a better approximation to a constant payoff require a different kind of contract. A natural candidate is the class of options.

An option is a contract that entitles the holder to buy or sell something at a future date, at his own discretion, at a prespecified price called the striking price. In the model we have considered there are four types of options markets, differing according to the date at which they equilibrate and the item to which the option represents a potential claim:

Type I: option for the commodity, traded at the intermediate date.

Type II: option for the commodity, traded at the prior date.

Type III: option for a contract (for unconditional delivery) on the intermediate market, traded at the prior date.

Type IV: (when type I options exist) an option for an option of type I deliverable at the intermediate date, traded at the prior date.

We will assume that if a type of options market exists, it is active at all striking prices and for rights to both purchase and sell. We maintain throughout the basic assumption that the equilibrium price of any contract is equal to its actuarialy fair value. Note that an unconditional contract of the type considered in the previous sections can be viewed as the right to buy at the zero striking price, or to sell at a striking price that is almost surely above the final spot price. (If the final spot price cannot be bounded above with probability one, then such options to sell can only approximate the sale of a contract for sure future delivery.) Because of this, it suffices to think of the market structure as consisting of options contracts only.

The basic results of this section can be summarized by the statement that if there are options markets in place of unconditional futures markets at each of the two trading dates, then an improvement in the information structure can never be

[9] Note also that the random variable attainable under $\mathscr{S}'$ might not be the result of any strategy under $\mathscr{S}$. See the discussion above under "Remarks on Proof."

detrimental. Two points should be noted: First, options markets provide far less opportunity for mitigating risks than would a full system of contingent contracts. Trades can be conditioned on the realization of $p(\cdot)$ to some extent, but not on $v(\cdot)$. For example, whatever options contracts are held, it will always be true that the net profit from trade depends only on x and p; for a fixed x there may be two points θ_1 and θ_2 in Θ where $p(\theta_1)=p(\theta_2)$ but $v(\theta_1)$ is very different from $v(\theta_2)$, and final wealth would necessarily be unequal in those cases. Second, the market structure with options contracts shares with the unconditional futures market system the property that prices at the intermediate date depend on the structure of information. Nevertheless, the proof of the basic result mentioned above will proceed by showing that, under some very mild conditions, the feasible set of payoff distributions is actually expanded by an improvement in information.

Let us examine the set of feasible payoff patterns in more detail. To do so, we need to know the profit or loss potentially available from trading each kind of option at each striking price. The prices of each of the four option contracts depend on whether it is a contract to buy or to sell, on the striking price and, for type I options, which are the only ones traded at the intermediate date, on the realized set $S \in \mathcal{S}$.

These prices are given as follows:

Type I:

$$\rho_+^I(q; S) = E_{\theta|S} \max(0, p(\theta)-q), \tag{7.1}$$

$$\rho_-^I(q; S) = -E_{\theta|S} \min(0, p(\theta)-q),$$

where + indicates an option to buy, − an option to sell, q is the striking price, and S is the observed event.

Type II:

$$\rho_+^{II}(q) = E_\theta \max(0, p(\theta)-q), \tag{7.2}$$

$$\rho_-^{II}(q) = -E_\theta \min(0, p(\theta)-q).$$

Type III:

$$\rho_+^{III}(q) = E_S \max(0, E_{\theta|S}\, p(\theta)-q), \tag{7.3}$$

$$\rho_-^{III}(q) = -E_S \min(0, E_{\theta|S}\, p(\theta)-q).$$

Type IV:

$$\rho_{+,(+,q')}^{IV}(q) = E_S \max(0, (E_{\theta|S} \max(0, p(\theta)-q'))-q) \tag{7.4}$$

is the price of an option to buy at the price q, an option to buy at the intermediate date with striking price q'. Other buy/sell permutations within type IV can be handled similarly.

Note that, just as the existence of options of type II at all striking prices removes the necessity for a separate treatment of unconditional trading at the prior date, the presence of options of type IV removes the relevance of those of type III.

We will now study how these options contracts can be used to mitigate the risks facing the agent. It will be useful to introduce the notion of a *profit pattern* which is a function

$$\eta : \Theta \times X \to \mathbb{R}$$

that gives the profit (or loss) obtained by the chosen combination of future contracts, as a function of the actual (θ, x) that is realized. We will describe the set of profit patterns that are attainable by giving some necessary conditions that they must satisfy and then showing that any profit pattern compatible with these requirements can in fact be realized by options trading.

An *options trading plan* is a description of the actions taken at the initial date, and the actions planned at the intermediate date depending on the available information. We will consider two types of market structures. Either type II or type III options will be traded at the initial date. Type I options will be traded at the intermediate date. We will neglect type IV options, although it will be seen that in fact they are not necessary.

An options trading plan must specify the net trade of each type of option at each striking price. Because the striking price is a continuous variable (there is a continuum of contracts) the appropriate way to deal with this spectrum of trading possibilities is to define *signed measures* over the real line, which give the "density" of contracts traded at various striking prices. For example, if J is an open interval of striking prices, then $z_+^{II}(J)$ $(z^{II}(J))$ is the number of options contracts of type II bought (sold) at striking prices in J. (Of course there is nothing to stop z_+^{II} or z_-^{II} from having point-masses at some $q \in \mathbb{R}$. In this case $z_+^{II}(J)$ will not go to zero as J converges downward to the point q.)

Type I options are not purchased until the intermediate date. The options trading plan must give a pair of signed measures $z_{+,S}^{I}$, $z_{-,S}^{I}$ for each $S \in \mathscr{S}$, the interpretation being that these are the trades to be executed when S is realized. Notice that the options of type II are in fact effectively converted into options of type I at the intermediate date. We adopt the convention that $z_{+,S}^{I}$, $z_{-,S}^{I}$ are the total amounts of type I options held after trading at that time.

In summary, an options trading plan is given by a vector of signed measures

$$(z_+^{II}, z_-^{II}, (z_{+,S}^{I}, z_{-,S}^{I})_{S \in \mathscr{S}})$$

when options of type I and II constitute the market structure. When the market structure is types I and III, then we have

$$(z_+^{III}, z_-^{III}, (z_{+,S}^{I}, z_{-,S}^{I})_{S \in \mathscr{S}})$$

with the same interpretation. Note that the definition of type III options causes them to yield a definitive return at the intermediate date rather than converting them into a type I option. Thus it is appropriate to view $z_{+,S}^{I}$, $z_{-,S}^{I}$ as actual net purchases at the intermediate date. This will become clear shortly when the reckoning of net profits is described.

Because of the conventions adopted above, each of the types of options contracts can be viewed as a single-period security. That is, the value of type II and III options are determined at the intermediate date according to the realization of S and the value of type I contracts are determined at the final date according to the realization of p.

For type I these values are given by

$$\omega^{I}_{+,q}(\theta) = \max(0, p(\theta) - q),$$
$$\omega^{I}_{-,q}(\theta) = -\min(0, p(\theta) - q), \tag{7.5}$$

where $\omega^{I}_{+,q}$ is the value of a contract to buy at a striking price of q.

For type II,

$$\omega^{II}_{+,q}(S) = E_{\theta|S} \max(0, p(\theta) - q) = \rho^{I}_{+}(q; S),$$
$$\omega^{II}_{-,q}(S) = -E_{\theta|S} \min(0, p(\theta) - q) = \rho^{I}_{-}(q; S). \tag{7.6}$$

For type III,

$$\omega^{III}_{+,q}(S) = \max(0, E_{\theta|S}p(\theta) - q),$$
$$\omega^{III}_{-,q}(S) = -\min(0, E_{\theta|S}p(\theta) - q). \tag{7.7}$$

We shall not, in fact, need the explicit expressions for type IV's valuations at the intermediate date.

We can compute the net profit from holding a unit of each type of options contract by simply taking the difference between its price (ex ante), given in (7.1)–(7.3) and its value ex post, given by (7.5)–(7.7).

Thus we have, for example in type I,

$$\zeta^{I}_{+,q}(S, \theta) = \max(0, p(\theta) - q) - E_{\theta|S} \max(0, p(\theta) - q) \tag{7.8}$$

as the net profit resulting on an option to buy the commodity, at a striking price q, executed after S is observed when θ is realized.

For type II we have a net profit that depends on the realized value of S, for example

$$\zeta^{II}_{+,q}(S) = E_{\theta|S} \max(0, p(\theta) - q) - E_{\theta} \max(0, p(\theta) - q) \tag{7.9}$$

and for type III we have, for example,

$$\zeta^{III}_{+,q}(S) = \max(0, E_{\theta|S}p(\theta) - q) - E_S \max(0, E_{\theta|S}p(\theta) - q). \tag{7.10}$$

We now come to a statement of some necessary conditions to be fulfilled by a feasible profit pattern. Whether the market structure is I and III or I and II, it is clear from (7.8)–(7.10) that the profit pattern $\eta(\theta, x)$ can be decomposed into the profit from trading contracts on the prior market (II or III) and those from the trading plan in the intermediate market (I). The former depend on x, but only through the realized S. The latter depend on S and θ, but for each S they vary only with the value of $p(\cdot)$. Therefore feasible profit patterns satisfy the necessary condition that

(7.11) $\eta(\theta, x)$ can be written as the composition of two functions $\eta_2(\eta_1(\theta, x))$ where $\eta_1 : \Theta \times X \to \mathbb{R} \times \mathcal{S}$ is defined by $\eta_1(\theta, x) = p(\theta)$, S_x such that $x \in S_x \in \mathcal{S}$.[10]

A second necessary condition is obvious from the observation that the net profit functions are continuous in $p(\cdot)$:

(7.12) The function $\eta_2 : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ is continuous in its first argument.

Finally, since each options contract is clearly actuarily fair, so must be the net profit function η. Thus,

(7.13) $E\eta(\theta, x) = 0$.

THEOREM: *Let* $\{E_{\theta|S}p(\theta)\}_{S \in \mathcal{S}}$ *be a set of distinct numbers. Then* (7.11)–(7.13) *are jointly sufficient for a function* $\eta : \Theta \times X \to \mathbb{R}$ *to be an attainable profit pattern, if options of type I and III are tradeable.*

PROOF: We will construct the function η in a straightforward manner. First compute $E_{\theta|S}\eta(\theta, x)$ for each $S \in \mathcal{S}$, and define this to be η_S. By using options of type III only we can make the net profit as a function of S duplicate η_S. The net profit attained by buying a unit of options to buy at striking price q can be characterized as a function of q and of $E_{\theta|S}p(\theta)$, as those functions that are

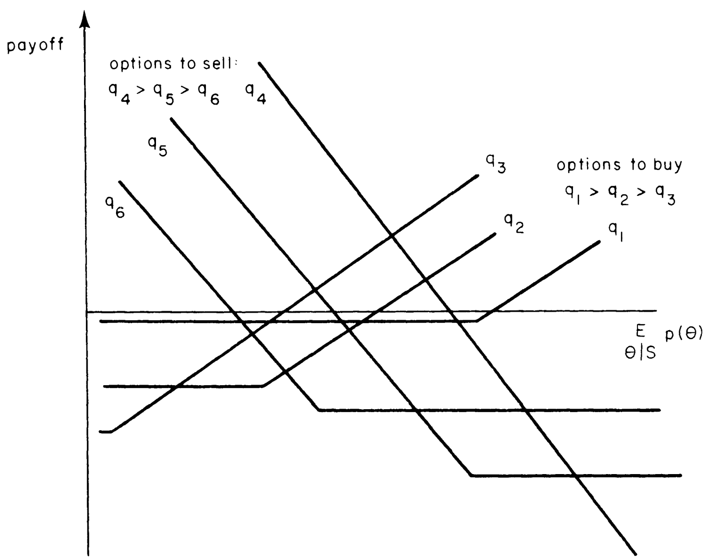

FIGURE 1.—Net profits on options contracts for various striking prices.

[10] Another way of saying this is that η is $\mathcal{P} \times \mathcal{S}$ measurable, where $\mathcal{P}$ is the algebra on Θ generated by $p(\cdot)$.

constant for $E_{\theta|S}p(\theta) < q$ and equal to $E_{\theta|S}p(\theta)$ plus a constant for $E_{\theta|S}p(\theta) \geqslant q$. These constants are determined by the condition that all contracts have mean zero and that the payoff be continuous in $E_{\theta|S}p(\theta)$. Options to sell at a striking price of q are similar, but the region of constant payoff is shifted to the right half-line.

Any linear combinations of these payoff functions is attainable by an options trading plan that specifies z_+^{III} and z_-^{III} as weights. It is easy to see, therefore, that by continuing z_+^{III} and z_-^{III} to be concentrated on finitely many points, one can obtain any (continuous) piecewise linear function of $E_{\theta|S}p(\theta)$ with mean zero as a payoff. Because the points $\{E_{\theta|S}p(\theta)\}_{S\in\mathcal{S}}$ are distinct, the payoff function can be fitted to the specified values of η_S when S is a finite partition. (If $E_{\theta|S_1}p(\theta) = E_{\theta|S_2}p(\theta)$, then of course $\eta_{S_1} = \eta_{S_2}$ would be implied, and thus a profit pattern that violated this condition but still satisfied (7.11)–(7.13) would be unattainable.) More generally, if $\mathcal{S}$ has an infinite number of members, an arbitrary function η_S can be approximated to any degree of accuracy by measures z_+^{III}, z_-^{III} concentrated on finitely many points, and their weak limit, which may be non-atomic, will attain the desired η_S pattern.

To complete the construction of η we need to achieve $\eta(\theta, x) - \eta_S$ as the payoff to the holdings of type I options contracts given the realization of S. Since $E_{\theta|S}\eta(\theta, x) - \eta_S = 0$ by construction, this amounts to showing that any actuarily fair payoff that varies only with p and is continuous in that variable can be attained by a combination of type I options. As in the case of type III above, we see from (7.8) that type I options have payoffs that are constant on some half-line and linear in $p(\theta)$ on the complementary half-line. Finite weighted combinations allow arbitrary continuous piecewise linear functions, and any continuous function can be reached by a limiting procedure. *Q.E.D.*

The pioneering paper by Ross [**21**] shows that the conditions of this theorem are likely to be true in general, and that the complete markets outcome could be expected in the presence of options, even though trades can be made only uncontingently. Ross's theorem applies only to the case of a finite set of states and observations, which might not be a good approximation to a more complex reality.

In some cases when $\mathcal{S}$ is a partition with an infinite set of elements and when Θ is multidimensional, the hypothesis that $\{E_{\theta|S}p(\theta)\}_{S\in\mathcal{S}}$ is a set of distinct numbers will not be valid. The following theorem applies to this situation.

THEOREM: *Let $\{p(\cdot) - E_{\theta|S}p(\theta)\}_{S\in\mathcal{S}}$ be a set of random variable whose distributions are distinct for different members of $\mathcal{S}$. Then* (7.11)–(7.13) *are jointly sufficient for a function $\eta : \Theta \times X \to \mathbb{R}$ to be an attainable profit pattern, if options of types I and II are tradeable.*

PROOF: The proof follows the lines of the previous theorem. Here, because options of type II allow a different payoff to be attained for different values of $S \in \mathcal{S}$ even though $E_{\theta|S}p(\theta)$ is the same, the hypothesis that the means be distinct can be weakened to the statement that the distributions be distinct.

These theorems provide us with the principle conclusion of this section.

THEOREM: *Let the set of attainable profit patterns be characterized by* (7.11)–(7.13), *for two information structures* $\mathcal{S}$ *and* $\mathcal{S}'$, *where* $\mathcal{S}$ *refines* $\mathcal{S}'$. *Then every agent would prefer* $\mathcal{S}$ *to* $\mathcal{S}'$.

PROOF. Let η' be attainable under $\mathcal{S}'$. Then η' satisfies (7.11)–(7.13) for $\mathcal{S}'$. For $\mathcal{S}$ we can define

$$\eta_2(p, S) = \eta_2'(p, S')$$

where $S \subseteq S' \in \mathcal{S}$, since $\mathcal{S}$ refines $\mathcal{S}'$. Clearly the continuity of η_2' in its first argument is inherited by η_2, which thus satisfies (7.11) and (7.12) under $\mathcal{S}$. Therefore the set of attainable profit patterns can only expand. *Q.E.D.*

These arguments show that the objectionable assumptions of the Theorem in Section 6 enforced by the linear structure of profits from futures trading, can be avoided if options markets exist. The conditions that remain are very mild, but are not entirely without force. If Θ and $\mathcal{S}$ are finite then, following the line of argument in Green [**9**] or Radner [**20**], it is possible to show that the conditions needed in the theorems of this section hold generically with respect to variations in the underlying joint distribution μ on $\Theta \times X$ and the functional dependence of v and p on θ. However, when $\mathcal{S}$ has a continuum of elements, for example when it consists of cylinder sets over points in a subspace of a high dimensional Euclidean space, the condition needed in the first theorem of this section is surely not satisfied. Even the weaker condition of the second theorem may not hold, although this is still an open question. Nevertheless it may be hoped that, although these conditions can be violated for some pairs of elements in the partition $\mathcal{S}$, the set of such pairs is sufficiently "thin" in $\mathcal{S} \times \mathcal{S}$ that the conclusion is still generically valid. We hope to approach this question in future work.

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REFERENCES

[**1**] BLACKWELL, D.: "Comparison of Experiments," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, ed. by J. Neymann. Berkeley: University of California Press, 1951, pp. 93–102.

[**2**] BLACKWELL, D., AND M. A. GIRSHICK: *Theory of Games and Statistical Decisions.* New York: John Wiley and Sons, 1954.

[**3**] BRADFORD, D., AND H. KELEJIAN: "The Value of Information for Crop Forecasting in a Market System," *Review of Economic Studies*, 44 (1977), 519–531.

[**4**] ———: "The Value of Information for Crop Forecasting with Baysian Speculators: Theory and Empirical Results," *Bell Journal of Economics*, 9 (1978), 123–144.

[**5**] FEIGER, G.: "Speculation and Equilibrium: Comment," *Quarterly Journal of Economics*, 90 (1976), 677–688.

[6] HIRSHLEIFER, J.: "The Private and Social Value of Information and the Reward to Inventive Activity," *American Economic Review*, 61 (1971), 561–574.
[7] ———: "Speculation and Equilibrium: Information, Risk and Markets," *Quarterly Journal of Economics*, 89 (1975), 519–542.
[8] ———: "The Theory of Speculation under Alternative Regimes of Markets," *Journal of Finance*, 32 (1977), 975–999.
[9] GREEN, J.: "Information, Efficiency and Equilibrium," Discussion Paper No. 284, Harvard Institute of Economic Research, March, 1973.
[10] ———: "Temporal General Equilibrium in a Sequential Trading Model with Spot and Futures Transactions," *Econometrica*, 41 (1973), 1103–1124.
[11] ———: "On the Economics of Information with Incomplete Markets," in *Systemes Dynamiques et Modeles Economiques*, Colloques Internationaux du C.N.R.S. No. 259, Paris, 1977.
[12] "The Value of Information with Uncontingent Futures Trading," Discussion Paper No. 555, Harvard Institute of Economic Research, June, 1977.
[13] GREEN, J., AND N. STOKEY: "Two Representations of Information Structures and Their Comparisons," Institute for Mathematical Studies in the Social Sciences, Stanford University, Technical Report No. 271, September, 1978.
[14] HAYAMI, Y., AND W. PETERSON: "Social Returns to Public Information Services: Statistical Reporting of U.S. Farm Commodities," *American Economic Review*, 62 (1972), 119–130.
[15] MARSCHAK, J., AND K. MIYASAWA: "Economic Comparability of Information Systems," *International Economic Review*, 9 (1968), 137–174.
[16] MARSHALL, J. M.: "Private Incentives and Public Information," *American Economic Review*, 64 (1974), 373–390.
[17] MCGUIRE, C. B.: "Comparison of Information Systems," in *Decision and Organization*, ed. by C. B. McGuire and R. Radner. Amsterdam: North-Holland, 1972.
[18] MCKINNON, R. I.: "Futures Markets, Buffer Stocks and Income Stability for Primary Producers," *Journal of Political Economy*, 75 (1967), 844–861.
[19] RADNER, R.: "Competitive Equilibrium Under Uncertainty," *Econometrica*, 36 (1968), 31–58.
[20] ———: "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," Working Paper No. IP–245, University of California-Berkeley, July, 1977.
[21] ROSS, S.: "Options and Efficiency," *Quarterly Journal of Economics*, 90 (1976), 75–89.